Levinson's Theorem for Non-local Interactions in Two Dimensions

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Abstract

In the light of the Sturm-Liouville theorem, the Levinson theorem for the Schrödinger equation with both local and non-local cylindrically symmetric potentials is studied. It is proved that the two-dimensional Levinson theorem holds for the case with both local and non-local cylindrically symmetric cutoff potentials, which is not necessarily separable. In addition, the problems related to the positive-energy bound states and the physically redundant state are also discussed in this paper.

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1. Introduction

The Levinson theorem [1], an important theorem in the scattering theory, established the relation between the total number of bound states and the phase shifts at zero momentum. The Levinson theorem has been proved by several authors with different methods, and generalized to different fields [2-9]. Generally speaking, three main methods are used to prove the Levinson theorem. One [1] is based on the elaborate analysis of the Jost function. This method requires nice behavior of the potential. The second, based on the Green function method [5], expounds that the total number of the physical states, which is infinite, is proved to be independent of the potential and the number of the bound states is the difference between the infinite numbers of the scattering states without and with the potential. The third method proves the Levinson theorem by the Sturm-Liouville theorem [6-8]. This simple, intuitive method is easy to be generalized. Some obstacles and ambiguities, which may occur in other two methods, disappear in the third method. We have succeeded in dealing with the nonrelativistic and relativistic problems in two dimensions by this way [10-11].

The reasons why we write this paper are that, on the one hand the Levinson theorem in two dimensions has been studied in experiment [12] as well as in theory [13,10,11] because of the wide interest in lower-dimensional field theories and other modern physics [14-20], on the other hand the Levinson theorem for non-local interactions in two dimensions has never appeared in the literature.

This paper is organized as follows. In Sec. 2, we establish the Sturm-Liouville theorem for non-local interactions in two dimensions. The Levinson theorem for this case will be set up in Sec. 3. Some problems related to the positive-energy bound states and the physically redundant state will be studied in Sec. 4 and 5.

2. The Sturm-Liouville Theorem

Throughout this paper $\hbar = 1$ and the mass $\mu = 1/2$ are employed for simplicity. Consider the Schrödinger equation with a local potential $V(r)$ and a non-local potential $U(r, r')$, where both potentials are cylindrically symmetric

$$\begin{aligned} H\psi(r,\varphi) &= -\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)\psi(r,\varphi)+V(r)\psi(r,\varphi) \\ &\quad +\int U(r,r')\delta(\varphi'-\varphi)\psi(r',\varphi')r'dr'd\varphi \\ &= E\psi(r,\varphi). \end{aligned} \tag{1}$$

The mesonic theory of nuclear forces points out the interaction between two nucleons is local at great distances but becomes non-local if the two nucleons come close. To simplify the expression, we assume, following Martin [4], that the non-local potential $U(r, r')$ is real, symmetric, vanishing at large distance, and not too singular at the origin [21],

$$\begin{aligned} &U(r,r') = U(r',r) \\ &|r^2U(r,r')| \sim 0, \qquad \text{at} \quad r \sim 0, \\ &U(r,r') = 0, \qquad\qquad \text{when} \quad r \geq r_0. \end{aligned} \tag{2}$$

As usual, the local potential $V(r)$ should not be too singular at the origin and at infinity. We assume that $V(r)$ satisfies

$$\begin{array}{ll} |r^2V(r)| \sim 0, & \text{at } \ r \sim 0, \\ V(r) = 0, & \text{when } \quad r \geq r_0. \end{array} \tag{3}$$

The first condition is necessary for the nice behavior of the wave function at the origin [1], and the potential with the second condition is called a cutoff one, namely, it is vanishing beyond a sufficiently large radius r_0. It was proved that the tail of the local potential at infinity will not change the essence of the proof [10] if it decays faster than r^{-3} at infinity. Under this assumption, the integral range in (1) is, in fact, from 0 to r_0, and the equation in the region $[r_0, \infty)$ becomes that for the free particle.

Introduce a parameter λ for the potentials

$$V(r, \lambda) = \lambda V(r), \qquad U(r, r', \lambda) = \lambda U(r, r'). \tag{4}$$

As λ increases from zero to one, the potentials $V(r, \lambda)$ and $U(r, r', \lambda)$ change from zero to the given potentials $V(r)$ and $U(r, r')$, respectively.

Owing to the symmetry of the potentials, letting

$$\psi(r, \phi, \lambda) = r^{-1/2} R_{Em}(r, \lambda) e^{\pm im\phi}, \qquad m = 0, 1, 2, \ldots, \tag{5}$$

we obtain the radial equation

$$\begin{aligned} &\frac{\partial^2}{\partial r^2} R_{Em}(r, \lambda) + \left(E - V(r, \lambda) - \frac{m^2 - 1/4}{r^2} \right) R_{Em}(r, \lambda) \\ &\quad = \sqrt{r} \int U(r, r', \lambda) R_{Em}(r', \lambda) \sqrt{r'} dr', \end{aligned} \tag{6}$$

where $\pm m$ and E denote the angular momentum and energy, respectively. Since the radial function $R_{Em}(r, \lambda)$ is independent of the sign of the angular momentum, we only discuss the case with non-negative m.

We are going to solve the radial equation (6) in two regions $[0, r_0]$ and $[r_0, \infty)$, and match two solutions at r_0. Since the Schrödinger equation is linear, the wavefunction can be multiplied by a constant factor. Removing the effect of the factor, we only need one matching condition at r_0 for the logarithmic derivative of the radial function:

$$A_m(E, \lambda) \equiv \left\{ \frac{1}{R_{Em}(r, \lambda)} \frac{\partial R_{Em}(r, \lambda)}{\partial r} \right\}_{r = r_0 -} = \left\{ \frac{1}{R_{Em}(r, \lambda)} \frac{\partial R_{Em}(r, \lambda)}{\partial r} \right\}_{r = r_0 +}. \tag{7}$$

We now turn to the Sturm-Liouville theorem. Denote by $\overline{R}_{Em}(r, \lambda)$ a solution of (6) with the energy $\overline{E}$. Multiplying the equations for $R_{Em}(r, \lambda)$ and $\overline{R}_{Em}(r, \lambda)$ by $\overline{R}_{Em}(r, \lambda)$ and $R_{Em}(r, \lambda)$, respectively, and calculating their difference, we have

$$\begin{aligned} &\frac{\partial}{\partial r} \left(R_{Em}(r, \lambda) \frac{\partial \overline{R}_{Em}(r, \lambda)}{\partial r} - \overline{R}_{Em}(r, \lambda) \frac{\partial R_{Em}(r, \lambda)}{\partial r} \right) + (\overline{E} - E) R_{Em}(r, \lambda) \overline{R}_{Em}(r, \lambda) \\ &\quad = \sqrt{r} R_{Em}(r, \lambda) \int U(r, r', \lambda) \overline{R}_{Em}(r', \lambda) \sqrt{r'} dr' \\ &\qquad - \sqrt{r} \overline{R}_{Em}(r, \lambda) \int U(r, r', \lambda) R_{Em}(r', \lambda) \sqrt{r'} dr'. \end{aligned} \tag{8}$$

According to the boundary condition, both $R_{Em}(r,\lambda)$ and $\overline{R}_{Em}(r,\lambda)$ go to zero when r tends to zero. Integrating (8) over the variable r in the region $[0,r_0]$ and noting the symmetric property of $U(r,r')$, we have

$$\begin{aligned}&\frac{1}{\overline{E}-E}\left\{R_{Em}(r,\lambda)\frac{\partial\overline{R}_{Em}(r,\lambda)}{\partial r}-\overline{R}_{Em}(r,\lambda)\frac{\partial R_{Em}(r,\lambda)}{\partial r}\right\}_{r=r_o-}\\&\quad=-\int_0^{r_0}R_{Em}(r',\lambda)\overline{R}_{Em}(r',\lambda)dr'.\end{aligned}$$

Taking the limit, we obtain

$$\begin{aligned}\frac{\partial A_m(E,\lambda)}{\partial E}&\equiv\frac{\partial}{\partial E}\left(\frac{1}{R_{Em}(r,\lambda)}\frac{\partial R_{Em}(r,\lambda)}{\partial r}\right)_{r=r_0-}\\&=-R_{Em}(r_0,\lambda)^{-2}\int_0^{r_0}R_{Em}(r',\lambda)^2dr'<0.\end{aligned}\tag{9}$$

Similarly, from the boundary condition that when $E\le 0$ the radial function $R_{Em}(r,\lambda)$ tends to zero at infinity, we have

$$\frac{\partial}{\partial E}\left(\frac{1}{R_{Em}(r,\lambda)}\frac{\partial R_{Em}(r,\lambda)}{\partial r}\right)_{r=r_0+}=R_{Em}(r_0,\lambda)^{-2}\int_{r_0}^{\infty}R_{Em}(r',\lambda)^2dr'>0.\tag{10}$$

Therefore, when $E\le 0$, both sides of the matching condition (7) are monotonic with respect to the energy E. As energy increases, the logarithmic derivative of the radial function at r_0- decreases monotonically, but that at r_0+ increases monotonically. This is an expression for the Sturm-Liouville theorem [22].

3. The Levinson Theorem

The establishment of the Levinson theorem for the case with both local and non-local cylindrically symmetric potentials is similar to that for the case with only a local potential.

In solving the radial equation (6) in the region $[0,r_0]$, only one solution is convergent at the origin. Thus, for the given potentials the logarithmic derivative $A_m(E,\lambda)$ is determined in principle. For example, for free particle ($\lambda=0$) we have

$$R_{Em}(r,0)=\begin{cases}\sqrt{\dfrac{\pi kr}{2}}J_m(kr), & \text{when}\ \ E>0\ \ \text{and}\ \ k=\sqrt{E}\\ e^{-im\pi/2}\sqrt{\dfrac{\pi\kappa r}{2}}J_m(i\kappa r), & \text{when}\ \ E\le 0\ \ \text{and}\ \ \kappa=\sqrt{-E},\end{cases}\tag{11}$$

where the factor in front of the radial function $R_{Em}(r)$ is not important. The solution $R_{Em}(r,0)$ given in (11) is a real function. The logarithmic derivative at r_0- for $E\le 0$ is

$$A_m(E,0)\equiv\left\{\frac{1}{R_{Em}(r,0)}\frac{\partial R_{Em}(r,0)}{\partial r}\right\}_{r=r_0-}=\begin{cases}\dfrac{kJ'_m(kr_0)}{J_m(kr_0)}-\dfrac{1}{2r_0} & \text{when}\ \ E>0\\ \dfrac{i\kappa J'_m(i\kappa r_0)}{J_m(i\kappa r_0)}-\dfrac{1}{2r_0} & \text{when}\ \ E\le 0\end{cases}\tag{12}$$

In the region $[r_0, \infty)$, we have $V(r) = U(r, r') = 0$. For $E > 0$, there are two oscillatory solutions to (6). Their combination can always satisfy the matching condition (7), so that there is a continuous spectrum for $E > 0$.

$$\begin{aligned} R_{Em}(r,\lambda) &= \sqrt{\frac{\pi kr}{2}}\left\{\cos\eta_m(k,\lambda)J_m(kr) - \sin\eta_m(k,\lambda)N_m(kr)\right\} \\ &\sim \cos\left(kr - \frac{m\pi}{2} - \frac{\pi}{4} + \eta_m(k,\lambda)\right), \qquad \text{when} \quad r \longrightarrow \infty, \end{aligned} \tag{13}$$

where $N_m(kr)$ is the Neumann function. The phase shift $\eta_m(k,\lambda)$ is determined by the matching condition (7)

$$\tan\eta_m(k,\lambda) = \frac{J_m(kr_0)}{N_m(kr_0)} \cdot \frac{A_m(E,\lambda) - kJ'_m(kr_0)/J_m(kr_0) - 1/2r_0}{A_m(E,\lambda) - kN'_m(kr_0)/N_m(kr_0) - 1/2r_0}, \tag{14}$$

$$\eta_m(k) \equiv \eta_m(k,1), \tag{15}$$

where the prime denotes the derivative of the Bessel function, the Neumann function, and later the Hankel function with respect to their argument. Although the radial equation (6) in the region $[r_0, \infty)$ is independent of λ, the solution $R_{Em}(r,\lambda)$ and the phase shift $\eta_m(k,\lambda)$ do depend on λ through the matching condition (7).

The phase shift $\eta_m(k,\lambda)$ is determined from (14) up to a multiple of π due to the period of the tangent function. Levinson determined the phase shift $\eta_m(k)$ with respect to the phase shift $\eta_m(\infty)$ at the infinite momentum. For any finite potential, the phase shift $\eta_m(\infty)$ will not change and is always equal zero. Therefore, Levinson's definition for the phase shift is equivalent to the convention that the phase shift $\eta_m(k,\lambda)$ is determined with respect to the phase shift $\eta_m(k,0)$ for the free particle, where $\eta_m(k,0)$ is defined to be zero

$$\eta_m(k,0) = 0, \qquad \text{where} \quad \lambda = 0. \tag{16}$$

There is some ambiguity for $\eta_m(\infty)$ when a bound state with a positive energy occurs (see Sec. 4). However, as far as the Levinson theorem is concerned, the latter convention is more convenient. We prefer to use this convention where the phase shift $\eta_m(k)$ is determined completely as λ increases from zero to one. It is the reason why we introduce the parameter λ.

For $E \leq 0$ there is only one convergent solution at infinity

$$R_{Em}(r) = e^{i(m+1)\pi/2}\sqrt{\frac{\pi\kappa r}{2}}H^{(1)}_m(i\kappa r) \sim e^{-\kappa r}, \qquad \text{when} \quad r \longrightarrow \infty. \tag{17}$$

where $H^{(1)}_m(x)$ is the Hankel function of the first kind. Thus, the matching condition (7) is not always satisfied. When the matching condition (7) is satisfied, a bound state appears at this energy. It means that there is a discrete spectrum for $E \leq 0$.
From (17) we have

$$\begin{aligned} \left\{\frac{1}{R_{Em}(r,0)}\frac{\partial R_{Em}(r,0)}{\partial r}\right\}_{r=r_0+} &= \frac{i\kappa H^{(1)}_m(i\kappa r_0)'}{H^{(1)}_m(i\kappa r_0)} - \frac{1}{2r_0} \\ &= \begin{cases} (-m+1/2)/r_0 \equiv \rho_m & \text{when} \quad E \longrightarrow 0 \\ -\kappa \sim -\infty & \text{when} \quad E \longrightarrow -\infty. \end{cases} \end{aligned} \tag{18}$$

On the other hand, if $V(r) = U(r, r') = 0$, we obtain from (12)

$$\begin{aligned} A_m(E,0) &\equiv \left\{ \frac{1}{R_{Em}(r,0)} \frac{\partial R_{Em}(r,0)}{\partial r} \right\}_{r=r_0-} \\ &= \frac{i\kappa J'_m(i\kappa r_0)}{J_m(i\kappa r_0)} - \frac{1}{2r_0} = \begin{cases} (m+1/2)/r_0 & \text{when } E \longrightarrow 0 \\ \kappa \sim \infty & \text{when } E \longrightarrow -\infty. \end{cases} \end{aligned} \tag{19}$$

It is easy to see from (18) and (19) that as energy increases from $-\infty$ to 0, there is no overlap between two variant ranges of the logarithmic derivatives at two sides of r_0 such that there is no bound state when $\lambda = 0$ except for S wave where there is a half bound state at $E = 0$. The half bound state will be discussed at the end of this section.

If $A_m(0, \lambda)$ decreases across the value $\rho_m \equiv (-m+1/2)/r_0$ as λ increases, an overlap between two variant ranges of the logarithmic derivatives at two sides of r_0 appears. Since the logarithmic derivatives of the radial function at r_0- decreases monotonically as the energy increases, and that at r_0+ increases monotonically, the overlap means that there must be one and only one energy where the matching condition (7) is satisfied, i.e. a scattering state changes to a bound state.

As λ increases, a zero point in the zero energy solution $R_{0m}(r, \lambda)$ may comes through r_0. In this process $A_m(0, \lambda)$ may decrease to the negative infinity, jumps to the positive infinity, and decreases again, or vice versa. It is not a singularity. If $A_m(0, \lambda)$ decreases, through the jump at infinity, again across the value ρ_m, another bound state appears.

As λ increases from zero to one, each time $A_m(0, \lambda)$ decreases across the value ρ_m, a new overlap between the variant ranges of two logarithmic derivatives appears such that a scattering state changes to a bound state. Conversely, each time $A_m(0, \lambda)$ increases across the value ρ_m, an overlap between those two variant ranges disappears such that a bound state changes back to a scattering state. The number of bound states n_m is equal to the times that $A_m(0)$ decreases across the value ρ_m as λ change from zero to one, subtracted by the times that $A_m(0)$ increases across the value ρ_m. In the following, we will show from (14) that this number is nothing but the phase shift $\eta_m(0)$ at zero momentum divided by π.

It is easy to see from (14) that the phase shift $\eta_m(k, \lambda)$ increases monotonically as the logarithmic derivative $A_m(E)$ decreases

$$\left. \frac{\partial \eta_m(k,\lambda)}{\partial A_m(E,\lambda)} \right|_k = -\frac{8 r_0 \cos^2 \eta_m(k)}{\pi \left(2 r_0 A_m(E) N_m(k r_0) - 2 k r_0 N'_m(k r_0) - N_m(k r_0)\right)^2} \leq 0. \tag{20}$$

The phase shift $\eta_m(0, \lambda)$ is the limit of the phase shift $\eta_m(k, \lambda)$ as k tends to zero. Therefore, what we are interested in is the phase shift $\eta_m(k, \lambda)$ at sufficiently small momentum k, $k \ll 1/r_0$. For the small momentum k we obtain from (14)

$$\tan\eta_m(k,\lambda)\sim$$
$$\sim\begin{cases}\dfrac{-\pi(kr_0)^{2m}}{2^{2m}m!(m-1)!}\cdot\dfrac{A_m(0,\lambda)-(m+1/2)/r_0}{A_m(0,\lambda)-c^2k^2-\rho_m\left(1-\dfrac{(kr_0)^2}{(m-1)(2m-1)}\right)} & \text{when } m\geq 2\\ \dfrac{-\pi(kr_0)^2}{4}\cdot\dfrac{A_m(0,\lambda)-3/(2r_0)}{A_m(0,\lambda)-c^2k^2-\rho_1\left(1+2(kr_0)^2\log(kr_0)\right)} & \text{when } m=1\\ \dfrac{\pi}{2\log(kr_0)}\cdot\dfrac{A_m(0,\lambda)-c^2k^2-\rho_0\left(1-(kr_0)^2\right)}{A_m(0,\lambda)-c^2k^2-\rho_0\left(1+\dfrac{2}{\log(kr_0)}\right)} & \text{when } m=0.\end{cases}\tag{21}$$

In addition to the leading terms, we include in (21) some next leading terms, which is useful only for the critical case where the leading terms are canceled with each other.

First of all, it can be seen from (21) that $\tan\eta_m(k,\lambda)$ tends to zero as k goes to zero, i.e. $\eta_m(0,\lambda)$ is always equal to the multiple of π. In other words, if the phase shift $\eta_m(k,\lambda)$ for a sufficiently small k is expressed as a positive or negative acute angle plus $n\pi$, where n is an integer, its limit $\eta_m(0,\lambda)$ is equal to $n\pi$. It means that $\eta_m(0,\lambda)$ changes discontinuously. By the way, in three dimensions, the phase shift at zero momentum of S wave may have an additional $\pi/2$ when the half bound state occurs.

Secondly, since the phase shift $\eta_m(k,\lambda)$ increases monotonically as the logarithmic derivative $A_m(E,\lambda)$ decreases, the phase shift at zero momentum $\eta_m(0,\lambda)$ will jump by π if $\tan\eta_m(k,\lambda)$ at the sufficiently small k changes sign from positive to negative as $A_m(E,\lambda)$ decreases, and vice versa. When λ changes from zero to one continuously, each time $A_m(0,\lambda)$ decreases from near and larger than the value ρ_m to smaller than that value, the denominator in (21) changes sign from positive to negative and the remaining factor keeps positive, such that the phase shift at zero momentum $\eta_m(0,\lambda)$ jumps by π. Conversely, each time $A_m(0,\lambda)$ increases across the value ρ_m, the phase shift at zero momentum $\eta_m(0,\lambda)$ jumps by $-\pi$. Therefore, the phase shift $\eta_m(0)/\pi$ is just equal to the times $A_m(0,\lambda)$ decreases across the value ρ_m as λ increases from zero to one, subtracted by the times $A_m(0,\lambda)$ increases across that value. Therefore, we proved the Levinson theorem for the Schrödinger equation in two dimensions for non-critical cases:

$$\eta_m(0)=n_m\pi.\tag{22a}$$

Thirdly, we should pay some attention to the case of $m=0$. When $A_0(0,\lambda)$ decreases across the value $\rho_0=1/(2r_0)$, both the numerator and denominator in (21) change signs, but not spontaneously because the next leading terms in the numerator and denominator in (21) are different. It is easy to see that the numerator changes sign first, and then the denominator changes sign, i.e. $\tan\eta_0(k,\lambda)$ at small k changes firstly from negative to positive, then to negative again such that $\eta_0(0,\lambda)$ jumps by π. Similarly, when $A_m(0,\lambda)$ increases across the value ρ_0, $\eta_0(0,\lambda)$ jumps by $-\pi$.

When $\lambda=0$ and $m=0$, the numerator in (21) is equal to zero, the denominator is positive, and the phase shift $\eta_0(0)$ is defined to be zero. If $A_0(E)$ decreases when λ increases from zero, the numerator becomes negative firstly, and then the denominator

changes from positive to negative such that the phase shift $\eta_0(0)$ jumps by π and simultaneously a new bound state appears.

Finally, we turn to discuss the critical cases where a half bound state occurs. If the logarithmic derivative $A_m(0,1)$ is equal to the value ρ_m, the following solution with zero energy in the region $[r_0,\infty)$ will match this $A_m(0,1)$ at r_0

$$R_{0m}(r,1) = r^{-m+1/2}.$$

It is a bound state when $m \geq 2$, but called a half bound state when $m = 1$, and 0. A half bound state is a zero energy solution of the Schrödinger equation which is finite but does not decay fast enough at infinity to be square integrable. We are going to discuss the critical case where $A_m(0,\lambda)$ decreases (or increases) and reaches, but not across, the value ρ_m as λ increases from a value, a little smaller than one, to one. For definiteness, we discuss the case where $A_m(0,\lambda)$ decreases and reaches the value ρ_m as λ increases to one. In this case a new bound state with zero energy appears for $m \geq 2$, but does not appear for $m = 1$ and 0. We need to check whether or not the phase shift $\eta_m(0)$ increases an additional π.

It is evident to find that the denominator in (21) for $m \geq 2$ has changed the sign from positive to negative as $A_m(0)$ decreases and reaches the value ρ_m, i.e. the phase shift $\eta_m(0)$ jumps by π and simultaneously a new bound state of zero energy appears.

For $m = 0$ the next leading term of the denominator in (21) is positive and larger than the term $-c^2k^2$, such that the denominator does not change sign, i.e. the phase shift $\eta_m(0)$ does not jump. It meets the fact that no new bound state appears.

For $m = 1$ the next leading term of the denominator in (21) is negative such that the denominator does change sign, i.e. the phase shift $\eta_m(0)$ jumps by π as $A_m(0)$ decreases and reaches the value ρ_1. However, in this case no new bound state appears simultaneously.

The discussion for the cases where $A_m(0)$ increases and reaches the value ρ_m is similar. Therefore, Levinson's theorem (22a) holds for the critical cases except for $m = 1$. In the latter case, Levinson's theorem for the Schrödinger equation with both local and non-local interactions in two dimensions becomes:

$$\eta_m(0) = (n_m + 1)\,\pi, \qquad \text{when} \quad m = 1, \quad \text{and a half bound state occurs.} \tag{22b}$$

As discussed above, it is found that the Levinson theorem holds without any modification for the case where a non-local potential is included.

4. Positive-energy bound states

It is well known that, in the case with only a local interaction, the wavefunction and its first derivative would never vanish at the same point except for the origin, so there is no positive-energy bound state. However, in the case with a non-local interaction, Martin showed that the solution with an asymptotic form is not unique when the potential satisfies some conditions [4], i.e. there exists the positive-energy bound state with a vanishing asymptotic form. If a small perturbative potential is added such that the non-local potential satisfies the conditions, the positive-energy bound state will appear and the phase shift at this energy increases rapidly by almost π. This can be seen explicitly in the examples given by Martin [4] and Kermode [23].

It was pointed out by Kermode that the inverse tangent function is not single-valued and it is physically more satisfactory to include a jump by π to the phase shift at the energy E_0, where a positive-energy bound state occurs. Martin and Chadan [4,21] defined the phase shift to be continuous even at E_0 so that an additional π will be included into $\delta(0)-\delta(\infty)$ for each positive-energy bound state. This is their reason to modify Levinson's theorem by the term $\sigma\pi$ where σ denotes the number of positive-energy bound state. But according to the viewpoint of Kermode, no modification to the Levinson theorem is required.

However, the phase shift at zero energy in our convention doesn't change, no matter which viewpoint is used, i.e. no matter whether the phase shift jumps or not at the energy with a positive-energy bound state. Therefore, the Levinson theorem (22) holds for the cases where positive-energy bound states may occur.

5. Redundant state

The resonating group model of the scattering of nuclei, or other composite systems, derives an effective two-body interaction in which a non-local potential appears. There are some physically redundant states which describe Pauli-forbidden states for the compound system, and the physical two-body states must be orthogonal to these redundant states [24]. In the case of three dimensions, Saito [25], Okai, et al [26] and Englefield-Shoukry [27] proposed a simple non-local term which guarantees that the required orthogonality, and verified that it was a good representation of the interactions. If there is just one redundant state represented by the real normalized wavefunction $U(r)$, then the two-dimensional Saito's equation is

$$\begin{aligned}&\frac{d^2}{dr^2}R_{Em}(r)+\left(E-V(r)-\frac{m^2-1/4}{r^2}\right)R_{Em}(r)\\&\quad=U(r)\int_0^\infty U(s)\left(\frac{d^2}{ds^2}-V(s)-\frac{m^2-1/4}{s^2}\right)R_{Em}(s)ds,\\&\int_0^\infty U^2(s)ds=1,\end{aligned}\tag{23}$$

and

$$E\int_0^\infty U(r)R_{Em}(r)dr=0.\tag{24}$$

The solution of (23) satisfies the orthogonality constraint except for that of zero energy. Saito's non-local potential is separable.

If the Schrödinger equation with only a local potential $V(r)$ has a bound state with a negative $-\mathcal{E}<0$, the corresponding wavefunction is denoted by $\psi(r)$:

$$\begin{aligned}&\frac{d^2}{dr^2}\psi(r)-\left(V(r)+\frac{m^2-1/4}{r^2}\right)\psi(r)=\mathcal{E}\psi(r),\\&\int_0^\infty\psi(r)^2dr=1.\end{aligned}\tag{25}$$

It is obvious that $U(r)=\psi(r)$ satisfies (23) with zero energy. Therefore, it is the so-called physically redundant state. As far as a mathematical equation (23) is concerned, the redundant state is one of the bound states with zero energy.

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